# Shock-expansion theory and simple wave perturbation

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The calculation by shock-expansion theory of supersonic aerofoil flow fields with varying entropy is simplified by assuming the flow behind the curved attached shock to be a small perturbation of a simple wave. A characteristic perturbation method is used.

## 1. Introduction

Shock-expansion theory was used by Epstein (1931) to determine surface pressures on an aerofoil behind a curved shock wave. In this theory the assumption is made that the flow at the surface is the same as in a simple wave, or Prandtl–Meyer expansion, the level of entropy being taken as that behind an oblique shock wave determined by the aerofoil leading-edge angle. Weight has been given to the validity of this assumption, particularly in the case $\gamma = 1{\cdot}4$, by Mahoney & Skeat (1955) who show that reflected disturbances at the shock wave tend to be largely cancelled out when propagating back to the aerofoil surface owing to the effect of the entropy gradient behind the curved shock.

Eggers, Syvertson & Kraus (1953) extended the concept, in their 'generalized shock-expansion method', to enable the entire flow field to be calculated. In this method a similar assumption to that of Epstein is now made for every streamline in the flow field, since any streamline could be replaced by a solid aerofoil surface. The method is simpler than the full numerical method of characteristics and yet takes into account the entropy gradients in the flow field.

In the present paper the above method of calculating the flow field is further simplified by assuming that the reflected disturbances and entropy gradients in the flow field are small and hence that the entire flow field can be described by means of a small perturbation of a simple wave (at the entropy level behind the attached shock at the aerofoil leading edge). The justification for this is that since at low supersonic speeds it is known that reflected disturbances and entropy gradients can be neglected to a good approximation (Friedrichs 1948) the present method should be applicable at slightly higher Mach numbers, where such effects are now taken into account to first order. Moreover, it appears that at even higher Mach numbers (say up to $M = 4$ or 5), where there is a larger entropy increase through the attached curved shock, the entropy variation in the field and reflected disturbances may still be small (Mahoney & Skeat 1955), their squares negligible, and the present method applicable.

The advantage of the present method is that whereas both the full characteristics method and ordinary shock-expansion theory need two-dimensional numerical calculations in the form of a mesh in order to find the curved shock shape and flow field behind it, the small perturbation assumption allows the shock shape to be calculated by a one-dimensional numerical step-by-step technique and the flow field behind it to be filled in later if required. The method uses a characteristic perturbation technique having a parametric variable which is taken as constant on perturbed outgoing characteristics.

An alternative approach to the problem of finding perturbations of simple wave flows using characteristic variables has been described by Waldman & Probstein (1961), based on work by Mahoney (1955). Also, an analytical development of shock-expansion theory, including the calculation of shock shape, has been given by Meyer (1956). However, the co-ordinate systems used by these authors are different from those used in the present paper, leading to results of quite different form. The present method appears to be better adapted to the simple evaluation of numerical results.

## 2. Review of basic equations

We begin by reviewing the equations of motion for the two-dimensional, steady, supersonic flow of a perfect gas. In characteristic form:

$$d\alpha + \sin 2\mu \, d\Phi = 0 \quad \text{on} \quad dy/dx = \tan(\theta - \mu), \tag{1}$$

$$d\beta - \sin 2\mu \, d\Phi = 0 \quad \text{on} \quad dy/dx = \tan(\theta + \mu), \tag{2}$$

and

$$d\Phi = 0 \quad \text{on} \quad dy/dx = \tan\theta, \tag{3}$$

where

$$\alpha = \theta + t, \tag{4}$$

$$\beta = \theta - t, \tag{5}$$

$x$, $y$ are Cartesian position co-ordinates in the plane of the flow, $\theta$ is the local angle made by a streamline with the $x$-axis, $\mu$ is the local Mach angle, $t$ the Prandtl angle given by

$$t = \mu - \lambda \tan^{-1}(\lambda \tan\mu) + \tfrac{1}{2}\pi(\lambda - 1) \tag{6}$$

with $\lambda^2 = (\gamma+1)/(\gamma-1)$, $\gamma$ being the ratio of specific heats, and

$$\Phi = S/2\gamma(\gamma-1)\,c_v, \tag{7}$$

where $S$ is the specific entropy and $c_v$ the specific heat at constant volume.

We consider the flow past a sharp-nosed aerofoil with attached curved shock wave. Then

$$dy/dx = \tan(\theta + \mu)$$

is the equation of outgoing, nearly straight, characteristics from the aerofoil to the shock wave, and

$$dy/dx = \tan(\theta - \mu)$$

is the equation of curvilinear characteristics carrying disturbances 'reflected' from the shock wave.

## 3. Review of assumptions of shock-expansion theory

The basic assumption of shock-expansion theory, when used for calculating the flow field (Eggers *et al.* 1953), is that on each streamline the flow is the same as in a simple wave, or Prandtl–Meyer expansion. In a simple wave

$$\alpha = \text{const.} \tag{8}$$

throughout the flow field. So the assumption can be written in the form

$$d\alpha = 0 \quad \text{on} \quad dy/dx = \tan\theta. \tag{9}$$

It is easily seen that equation (9) is not exactly compatible with the basic equations in general. For, with equation (3), equation (9) implies a functional relationship between $\alpha$ and $\Phi$, and taken with equation (1) this would imply that $\mu$ did not vary along streamlines. In the exact flow field lines of constant $\alpha$ make a small angle with streamlines $dy/dx = \tan\theta$ (this follows from the work of Mahoney & Skeat 1955) but cannot in general exactly coincide with them. However, if equation (9) is used and equation (1) is dropped, a consistent set of equations is obtained, from which the approximate flow field can be calculated.

Eggers *et al.* (1953) show that an alternative, but equivalent, basis for shock-expansion theory is to use the relationship:

$$d\theta = 0 \quad \text{on} \quad dy/dx = \tan(\theta+\mu) \tag{10}$$

instead of equation (9). It follows from equations (1), (2) and (3) that when equation (9) is a good approximation, equation (10) is equally so. Under these conditions lines of constant $\theta$ make a small angle with outgoing characteristics $dy/dx = \tan(\theta+\mu)$ in the exact flow field but cannot in general exactly coincide with them. The alternative basis of shock-expansion theory, then, is to use the consistent set of equations obtained by assuming equation (10) and dropping, as before, equation (1). This latter approach is the one to be followed, for reasons of algebraic convenience, in the following sections.

## 4. Simple wave perturbation

The basis of the present method is to obtain the flow field behind the shock wave, in the form of a small perturbation of the simple wave defined by the boundary conditions at the aerofoil surface and the entropy level behind the attached shock at the aerofoil leading edge, by solving the exact equations (2) and (3) with the approximate equation (10) used as a subsidiary condition. The resulting equations are now derived and it is shown how the flow field behind the shock wave can be found parametrically. The numerical method for finding the shock-wave position, which completes the solution, is described in the following section.

The solution for the flow field behind the shock wave is expressed in terms of co-ordinates $(\psi_0, \eta)$, where $\psi_0$ is constant on the streamlines of the unperturbed simple wave, now assumed known, and $\eta$ is a parametric variable which is constant on the outgoing characteristics of the perturbed flow field (figure 1). Suffix zero is taken to denote the basic simple wave

$$\alpha_0 = \text{const.} \tag{11}$$

A co-ordinate $s$ is used to denote distance along the aerofoil measured from the leading edge O (figure 1). Then each characteristic, $\eta = \text{const.}$, is uniquely related by its point of origin F on the aerofoil surface (figure 1) to a corresponding characteristic, $\eta_0 = \text{const.}$, of the basic simple wave, giving the following relationships:

$$\eta = f(s), \tag{12}$$

$$\eta_0 = g(s), \tag{13}$$

$$\eta_0 = \eta_0(\eta). \tag{14}$$

Equation (2) can now be written in the form

$$\partial\beta(\psi_0, \eta)/\partial\psi_0 = \sin 2\mu(\psi_0, \eta)\,\partial\Phi(\psi_0, \eta)/\partial\psi_0. \tag{15}$$

Writing
$$\theta_2 = \theta_0 + \theta_1, \quad t_2 = t_0 + t_1, \tag{16}$$
and
$$\beta_2 = \beta_0 + \beta_1, \tag{17}$$

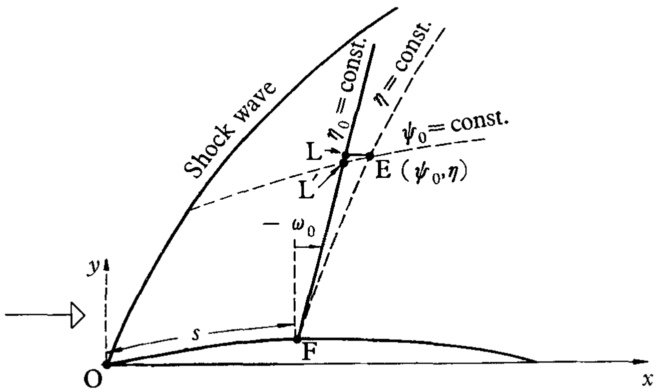


FIGURE 1. The co-ordinate system.

where the suffix unity denotes small perturbation terms, the linearized form of equation (15) is
$$\partial\beta_2(\psi_0, \eta)/\partial\psi_0 = \sin 2\mu_0\,\partial\Phi_1\{\psi_0, \eta_0(\eta)\}/\partial\psi_0, \tag{18}$$
where $\mu_0$ takes its value on the basic simple-wave characteristic $\eta_0 = \text{const.}$, $\eta_0$ being related to $\eta$ by equation (14). Similarly, the linearized forms of equations (10) and (3) are
$$\partial\theta_2(\psi_0, \eta)/\partial\psi_0 = 0 \tag{19}$$
and
$$d\Phi_1 = 0 \quad \text{on} \quad dy/dx = \tan\theta_0. \tag{20}$$

Using the boundary condition of zero perturbation at the aerofoil surface the solutions of the above equations give

$$\theta_2(\psi_0, \eta) = \theta_0\{\eta_0(\eta)\}, \tag{21}$$

$$t_2(\psi_0, \eta) = t_0\{\eta_0(\eta)\} - \sin 2\mu_0\,\Phi_1(\psi_0), \tag{22}$$

$$\Phi_1 = \Phi_1(\psi_0). \tag{23}$$

The function $\Phi_1(\psi_0)$ is determined by the entropy change at the shock wave. When the shock wave is known, equations (21) and (22) describe the flow field completely at the point $E(\psi_0, \eta)$ (figure 1), where $\eta$ is regarded as a parametric variable.

It remains to locate the point $E(\psi_0, \eta)$ in the flow plane. In general this would be done by expressing Cartesian co-ordinates $(x, y)$ in terms of the variables $(\psi_0, \eta)$. Here, however, it is more convenient from a computational point of view to locate the point E by finding the distance LE (figure 1) (taken as positive when E is downstream of L), parallel to the free stream, from E to the line $\eta_0 = \text{const.}$, where $\eta_0$ is given by equation (14). The position of E can then be found by an elementary construction.

The tangent at a point $(\psi_0, \eta)$ on the perturbed characteristic $\eta$ = const. makes an angle with the associated characteristic $\eta_0$ = const. of, to first order,

$$(\theta_2-\theta_0)+(\mu_2-\mu_0) = (\mu_2-\mu_0) = (dt/d\mu)_0^{-1}(t_2-t_0) = -(dt/d\mu)_0^{-1}\Phi_1(\psi_0)\sin 2\mu_0,$$

where equations (21) and (22) have been used. Integration then gives the length of the perpendicular from E on to $\eta_0$ = const., and hence

$$\mathrm{LE} = \operatorname{cosec}(\theta_0+\mu_0)\left(\frac{dt}{d\mu}\right)_0^{-1}\sin 2\mu_0\int_0^r \Phi_1(\psi_0)\,dr, \tag{24}$$

where $r$ is strictly the distance from F (figure 1) to the foot of the perpendicular from E on to $\eta_0$ = const., but to first order can be more conveniently replaced by the distance FL′ (figure 1) where L′ lies on the basic streamline $\psi_0$ = const. It then follows from Courant & Friedrichs (1948) that the stream function $\psi_0$ can be chosen so that

$$r = \psi_0[\cos\lambda^{-1}(\omega_0-\omega_*)]^{-\lambda^2}, \tag{25}$$

where $\omega_0 = \omega_0(\eta_0)$ is the angle between the $y$-axis and the outgoing characteristic (figure 1), $\omega_*$ referring to critical conditions. Equations (24) and (25) then give

$$\mathrm{LE} = \left\{\operatorname{cosec}(\theta_0+\mu_0)\left(\frac{dt}{d\mu}\right)_0^{-1}\sin 2\mu_0[\cos\lambda^{-1}(\omega_0-\omega_*)]^{-\lambda^2}\right\}\int_0^{\psi_0}\Phi_1(\psi_0)\,d\psi_0. \tag{26}$$

This completes the location of the point $\mathrm{E}(\psi_0, \eta)$, and the parametric solution for the flow field, when the shock-wave position is known. In the following section it is shown how these results can be used as the basis of a numerical method for calculating this shock-wave position.

## 5. Calculation of shock-wave position

The final step in the development of the method is to note that, although the term in braces in equation (26) refers to the properties of the basic simple wave on the characteristic $\eta_0$ = const. through L, it may be replaced to first order, since the factor outside the brackets is a small quantity, by its value on the characteristic of the basic simple wave F′E, through E (figure 2). The step-by-step shock-wave calculation (figure 2) now proceeds as follows.

A basic mesh consisting of the known streamlines and outgoing straight characteristics of the basic simple wave defined by the aerofoil shape, at the entropy level behind the oblique shock attached at the leading edge, is used. Suppose the shock-wave shape to have been calculated as far as the point E (figure 2) in $(n-1)$ steps. The $n$th step is constructed as follows. The approximate value of

$$\int_0^{\psi_0}\Phi_1(\psi_0)\,d\psi_0$$

is obtained from the steps already constructed. The value of the bracketed term in equation (26) is determined for the basic simple-wave characteristic F′E through E, as already described. Equation (26) then gives the distance LE (figures 1, 2), determining the basic characteristic FL, $\eta_0$ = const. Equation (21) then gives the flow deflection in the perturbed flow field at E, from which the new shock angle for the $n$th step is found. The corresponding value of $\Phi_1$ and the value of $d\psi_0$ (from equation (25)) for the $n$th step enable the new approximate value of

$$\int_0^{\psi_0} \Phi_1(\psi_0)\, d\psi_0$$

to be found. The process can now be repeated.

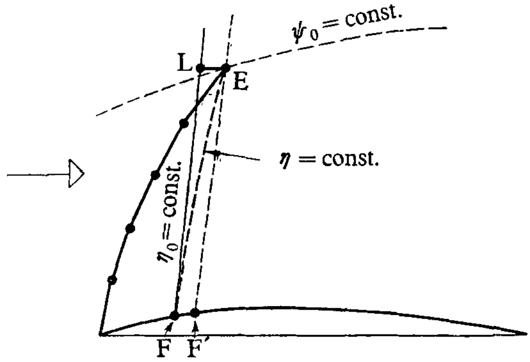


FIGURE 2. Step-by-step shock calculation.

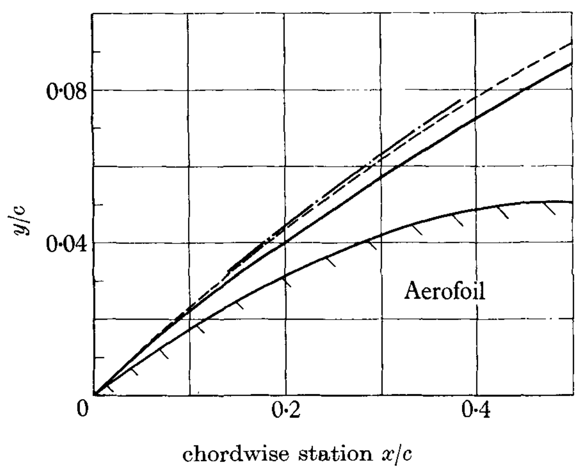


FIGURE 3. Shape of shock wave for 10 % thick biconvex aerofoil at $M_0 = \infty$. ——, Characteristics; – – – –, shock-expansion method; —·—·—, present method.

The advantage of the above method of shock-wave calculation over the usual shock-expansion theory is that it is a one-dimensional step-by-step construction, whereas the usual theory requires the numerical computation of a two-dimensional mesh over the whole flow field. In the present method the perturbed outgoing characteristics, and other details of the flow field behind the shock, can be filled in if required after the shock-wave calculation has been completed, as described in § 4.

As stated earlier, by taking the Mach number low enough, the basic assumptions (of small entropy perturbation and small reflected disturbances) become realized and a simple method is provided for calculating flow fields at least at

Mach numbers not much beyond the range in which the methods of Friedrichs (1948) apply. Unfortunately, there is a lack of published full shock-expansion solutions and of characteristics solutions at higher Mach numbers ($M = 4$ or 5), so a detailed check on the range of application of the present method has not been made. However, the shock-wave position calculated by the present method is compared in figure 3 with that calculated by full shock-expansion theory and by the method of characteristics (from Eggers *et al.* 1953) in the limiting case of infinite free-stream Mach number, taking $\gamma = 1{\cdot}4$. The excellent agreement with full shock-expansion theory in this case is surprising, and probably somewhat fortuitous. However, the method certainly gives sensible results in this particular case and thus may even be useful at high Mach numbers if approximate results are required with a minimum of numerical effort.

## 6. Conclusions

A method for calculating the flow field past a sharp-nosed aerofoil at supersonic speeds has been presented. The assumptions made include those of the usual shock-expansion theory and the extra assumption that the flow field behind the curved shock wave can be represented as a small perturbation of a simple wave, the basic level of entropy being taken as that behind the attached shock at the leading edge. The advantage of the method is that the shock-wave position can be calculated in a one-dimensional numerical step-by-step manner, instead of requiring a numerical computation over a two-dimensional mesh covering the whole flow field. The flow field behind the shock wave can easily be deduced if required.

This approach should give results as good as those of ordinary shock-expansion theory in the low-Mach-number range where entropy changes at the shock first become important (about $M = 3$), since at slightly lower Mach numbers the terms now regarded as small are negligible (Friedrichs 1948). For a particular aerofoil a comparison of calculated shock-wave position has been made with shock positions given by ordinary shock-expansion theory and full characteristics in the limiting case of infinite free-stream Mach number (and $\gamma = 1{\cdot}4$). The result suggests that the method may also be useful at high Mach numbers if approximate results are required with a minimum of numerical effort.

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